Shock waves on current-carrying metal thin films

R. Mark Bradley
Department of Physics, Colorado State University, Fort Collins, Colorado 80523, USA
(Received 10 November 2008; published 2 February 2009)

It is demonstrated that an electrical current can produce abrupt macroscopic steps that retain their shape as they propagate on the surface of a miscut single-crystal metal thin film. A simple kinematic theory is developed that shows why these shock waves form, and that gives their velocity. We then derive an approximate equation of motion that allows us to study the structure and interactions of shocks that propagate along the applied electric field, as well as the propagation and structure of oblique shocks. If the surface height is independent of the transverse coordinate, the approximate equation of motion reduces to the Burgers equation. In general, the equation of motion is the dissipative Zabolotskaya-Khokhlov equation.

DOI: 10.1103/PhysRevB.79.075403 PACS number(s): 66.30.Qa, 68.35.Ja, 05.45.−a

I. INTRODUCTION

When an electrical current passes through a conducting solid, collisions between the conduction electrons and the atoms at the surface lead to drift of these atoms. This phenomenon, which is known as surface electromigration (SEM), can cause the surface of a current-carrying solid to move and deform.[1–24] The free surface of a conducting solid moves in response to the electrical current flowing through the bulk, in much the same way that flow in the bulk of a fluid affects the motion of its surface. However, this analogy is not perfect: the boundary conditions are very different in the two problems and, due to its crystal structure, a conducting solid is anisotropic.

Two complementary approaches are used to model the motion of a free surface brought about by SEM. If the surface is vicinal and the temperature of the solid $\mathcal{T}$ is below the roughening temperature $\mathcal{T}_R$, step flow models are employed (see, for example, Refs. 1–6). In this type of model, the electrical current is taken to be independent of position. Step flow models are able to account for many aspects of the SEM-induced formation of step bunches on a vicinal surface of silicon.[1–6] On the other hand, if the surface of the conductor is far from being planar, the spatial variation in the electrical current must be taken into account. In these circumstances, a coarse-grained continuum description is used in which the surface height $h=h(x,y,t)$ and the mobility tensor of the surface atoms $M_{ij}=M_{ij}(\nabla h)$ are assumed to be continuous functions of their arguments.[7–24] Mahadevan and co-workers[17,18] used a continuum model to explain the genesis of the slit-shaped voids that can lead to electrical failure of single-crystal metal lines, for example. In general, the temperature $\mathcal{T}$ must be above the roughening temperature $\mathcal{T}_R$ if a continuum description is to be applied since, for $\mathcal{T}<\mathcal{T}_R$, facets may form on the surface of the conductor.

The continuum description of SEM leads to very rich nonlinear surface dynamics. The planar surface of a current-carrying single-crystal miscut film is either linearly stable or unstable, depending on the Miller indices of the surface. Let the z axis be normal to the undisturbed planar free surface of the conducting film and take the applied electric field to be in the x direction. In prior work on this problem, the surface height h was taken to be independent of the transverse coordinate y, i.e., the surface disturbance was assumed to be one dimensional.[7,11] If there is an instability, waves are formed with a wavelength that is determined by the competition between the stabilizing effect of surface diffusion and the destabilizing effect of the slope dependence of the adatom mobility. At first, the amplitude of the waves grows exponentially with time but as time passes nonlinear effects become increasingly important. The wave amplitude then starts to saturate and the wavelength begins to grow.[7,11]

Now consider the stable regime. If the amplitude of the initial disturbance is small enough, the equations of motion can be linearized and the amplitude of each Fourier component of the surface height simply decays exponentially in time.[7,11] The situation is more complex, however, if the amplitude of the initial disturbance is large enough for nonlinear effects to be appreciable. We will show that in this case abrupt macroscopic steps can develop on the surface, and that these remain their form as they propagate. Such a surface disturbance is called a *shock wave*.[25]

Shock waves are not to be confused with the step bunches that can form on a vicinal silicon surface.[26,27] It is true that surface electromigration is the physical mechanism responsible for the development of both shock waves and step bunches. However, shock waves form on stable surfaces above the roughening temperature; in contrast, step bunches are observed on unstable surfaces below the roughening temperature. Shock wave formation is an inherently nonlinear phenomenon and the leading-order nonlinearity comes from spatial variations in the electrical current. Step bunches, on the other hand, are the result of a linear instability and the electrical current is taken to be uniform in theories of their genesis.

In this paper, we will study the formation, structure, and interaction of shock waves on the surface of a current-carrying single-crystal miscut metal film.[28] To begin, we extend past work on the linear stability of the free surface to the general case in which the surface height h depends on both x and y. We then turn to the problem of shock formation, and so restrict our attention to the time evolution of stable surfaces. Initially, we develop a simple kinematic theory of one-dimensional shocks. This theory shows why shocks form and yields their velocity but reveals nothing about the internal structure of a shock because it omits smoothing effects. We then derive an approximate equation of motion that includes

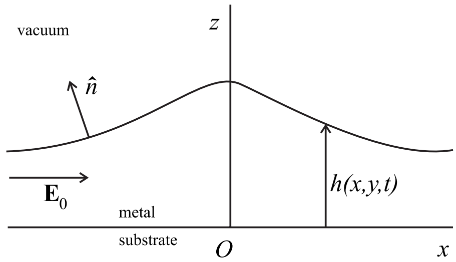

FIG. 1. The current-carrying metal thin film. The height of the free surface above the substrate, h, depends on x, y, and t. The outward-pointing unit normal to the free surface is $\hat{n}$, and the electric field far from the localized disturbance is $\mathbf{E}_0 = E_0\hat{x}$.

smoothing effects, and that applies when h varies slowly with y. This equation allows us to study the structure of one-dimensional shocks and their interactions, as well as the propagation and structure of oblique shocks, i.e., shocks whose direction of propagation makes a small angle with the x axis. If the surface height is independent of y, the approximate equation of motion reduces to the much-studied Burgers equation. The Burgers equation is the simplest partial differential equation that develops shocks in one dimension, is exactly solvable, and arises in many fields, including nonlinear acoustics, turbulence, thin-film deposition, and cosmology.[29] In the general case in which h depends on both x and y, the equation of motion is the dissipative Zabolotskaya-Khokhlov (DZK) equation,[30–32] which was first studied in nonlinear acoustics.

II. EQUATIONS OF MOTION

Consider a single-crystal metal film of uniform thickness h_0 deposited on the plane surface of an insulating substrate. We take the z axis to be normal to the substrate surface and locate the origin in this plane. The temperature $\mathcal{T}$ is well below the roughening temperature $\mathcal{T}_R$. We assume that the free surface of the film is miscut with equally spaced atomic steps parallel to the y direction, and that the crystal structure is invariant under the transformation $y \to -y$.

A potential difference is now applied across the film and the temperature is raised above the roughening temperature. The electric field within the film is $E_0\hat{x}$, where E_0 is a positive constant. Next, the upper surface of the film is disturbed. Since $\mathcal{T} > \mathcal{T}_R$, we may use a continuum description of the film in which the height of the film's surface above the substrate h is a continuous function of x, y, and t (Fig. 1). The free surface of the film will evolve in the course of time due to the effects of SEM and surface self-diffusion. For the sake of simplicity, we will assume that the applied current is held fixed and that it is high enough that the effects of SEM are much more important than those of capillarity. We will also assume that the mobility of the metal atoms is negligible at the metal-insulator interface so that the form of that interface remains planar for all time.

Let the outward-pointing unit normal to the upper film surface be $\hat{n}$. The electrical potential $\Phi = \Phi(x,y,z,t)$ satisfies the Laplace equation

$$\nabla^2 \Phi = 0 \tag{1}$$

within the film,[33] and is subject to the boundary condition $\hat{n} \cdot \nabla\Phi = 0$ on the upper surface and $\hat{z} \cdot \nabla\Phi = 0$ on the lower. Explicitly, we have

$$\Phi_z(x,y,h,t) = h_x \Phi_x(x,y,h,t) + h_y \Phi_y(x,y,h,t), \tag{2}$$

and

$$\Phi_z(x,y,0,t) = 0. \tag{3}$$

($f_x \equiv \partial f / \partial x$ for an arbitrary function f, and the subscripts y, z, and t are defined analogously.) If the initial disturbance is localized in the x direction, we also have

$$h(x,y,t) \to h_0 \tag{4}$$

for $x \to \pm\infty$ and, furthermore,

$$\Phi_x(x,y,z,t) \to -E_0, \tag{5}$$

and

$$\Phi_y(x,y,z,t) \to 0 \tag{6}$$

for $x \to \pm\infty$, arbitrary y, and $0 \leq z \leq h_0$.

An adatom at the free surface of the metal film is subject to the force $\mathbf{F} = q\mathbf{E}$, where q is the effective charge and $\mathbf{E} = -\nabla\Phi$ is the electric field. As a result, the atom drifts along the surface. Provided that the applied current is not extremely large, the atomic drift velocity is a linear function of the force. (This is certainly the case in all experiments performed to date.) If the metal were isotropic, the atom's drift velocity $\mathbf{v}$ would simply be proportional to $\mathbf{F}$, and the constant of proportionality would be the mobility. However, since the metal is a single crystal, the surface is anisotropic, and the mobility is a second-rank tensor M_{ij} which depends on the surface normal $\hat{n}$.

Let $\mathbf{R}(x,y,t) \equiv x\hat{x} + y\hat{y} + h(x,y,t)\hat{z}$ be an arbitrary point on the free surface of the metal. The vectors

$$\mathbf{R}_x \equiv \frac{\partial \mathbf{R}}{\partial x} = \hat{x} + h_x\hat{z} \tag{7}$$

and

$$\mathbf{R}_y \equiv \frac{\partial \mathbf{R}}{\partial y} = \hat{y} + h_y\hat{z} \tag{8}$$

form a basis for the plane tangent to the surface at $\mathbf{R}$, and so a generic vector $\mathbf{A}$ in the tangent plane may be written $\mathbf{A} = A_1\mathbf{R}_x + A_2\mathbf{R}_y$. The atomic drift velocity $\mathbf{v} = v_1\mathbf{R}_x + v_2\mathbf{R}_y$ depends linearly on the force $\mathbf{F} = F_1\mathbf{R}_x + F_2\mathbf{R}_y$, i.e., $v_i = M_{ij}F_j$, where the Einstein convention has been employed. The surface atomic current is $\mathbf{J} = \nu_s\mathbf{v}$, where ν_s is the number of mobile surface atoms per unit area.

We assumed that the crystal structure is unchanged by the reflection $y \to -y$. This means that the surface atomic current on the unperturbed surface cannot have a y component, and hence $M_{21}(0) = 0$. Also, because the adatoms must drift in the direction of the applied force, $M_{11}(0)$ must be positive. These observations will be needed later in the paper.

At the arbitrary surface point $\mathbf{R}(x,y,t)$, the electric field $\mathbf{E} = -\nabla\Phi = -(\Phi_x\hat{x} + \Phi_y\hat{y} + \Phi_z\hat{z})$ is tangent to the surface, and so $\mathbf{E}$ is a linear combination of $\mathbf{R}_x$ and $\mathbf{R}_y$. Explicitly, at the surface

$$-\mathbf{E} = \frac{1}{g}[(1+h_y^2)\Phi_x - h_x h_y \Phi_y + h_x \Phi_z]\mathbf{R}_x + \frac{1}{g}[(1+h_x^2)\Phi_y - h_x h_y \Phi_x + h_y \Phi_z]\mathbf{R}_y, \tag{9}$$

where $g=1+h_x^2+h_y^2$ is the determinant of the metric tensor. Since $\Phi=\Phi[x,y,h(x,y,t),t]$ at the surface, the total derivative of Φ with respect to x is $\partial_x\Phi=\Phi_x+\Phi_z h_x$. Similarly, $\partial_y\Phi=\Phi_y+\Phi_z h_y$. In terms of these total derivatives, Eq. (9) is

$$-\mathbf{E} = \frac{1}{g}[(1+h_y^2)\partial_x\Phi - h_x h_y \partial_y\Phi]\mathbf{R}_x + \frac{1}{g}[(1+h_x^2)\partial_y\Phi - h_x h_y \partial_x\Phi]\mathbf{R}_y. \tag{10}$$

Because $J_i=\nu_s v_i=\nu_s M_{ij}F_j=q\nu_s M_{ij}E_j$, the components of the surface atomic current are

$$J_1 = -\frac{q\nu_s}{g}\{M_{11}[(1+h_y^2)\partial_x\Phi - h_x h_y \partial_y\Phi] + M_{12}[(1+h_x^2)\partial_y\Phi - h_x h_y \partial_x\Phi]\}, \tag{11}$$

and

$$J_2 = -\frac{q\nu_s}{g}\{M_{21}[(1+h_y^2)\partial_x\Phi - h_x h_y \partial_y\Phi] + M_{22}[(1+h_x^2)\partial_y\Phi - h_x h_y \partial_x\Phi]\}. \tag{12}$$

If there is a net flow of atoms into or out of a surface element, the surface element will move. The sign of the normal velocity v_n will be chosen so that it is positive if the surface element moves in the $+\hat{n}$ direction. Explicitly,

$$v_n = -\Omega \, \mathrm{div}\,\mathbf{J}, \tag{13}$$

where Ω is the atomic volume and div $\mathbf{J}$ is the surface divergence[34] of the atomic current. Since the free surface of the metal film is in general curved, div $\mathbf{J}$ is not simply $\partial J_1/\partial x+\partial J_2/\partial y$; instead

$$\mathrm{div}\,\mathbf{J} = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial x}(\sqrt{g}J_1) + \frac{\partial}{\partial y}(\sqrt{g}J_2)\right]. \tag{14}$$

Combining Eqs. (11)–(14) with the result

$$h_t = \sqrt{g}v_n, \tag{15}$$

we obtain the equation of motion for the free surface:

$$\frac{1}{q\nu_s\Omega}\frac{\partial h}{\partial t} = \partial_x\left(\frac{1}{\sqrt{g}}\{M_{11}[(1+h_y^2)\partial_x - h_x h_y\partial_y] + M_{12}[(1+h_x^2)\partial_y - h_x h_y \partial_x]\}\Phi\right) + \partial_y\left(\frac{1}{\sqrt{g}}\{M_{21}[(1+h_y^2)\partial_x - h_x h_y \partial_y] + M_{22}[(1+h_x^2)\partial_y - h_x h_y \partial_x]\}\Phi\right). \tag{16}$$

On the right-hand side of this equation, $\Phi=\Phi(x,y,h(x,y,t),t)$, the total derivative with respect to x is denoted by ∂_x, and ∂_y is similarly defined. We remind the reader that the components of the mobility tensor M_{ij} depend on $\hat{n}$, or, equivalently, $M_{ij}=M_{ij}(h_x,h_y)$. Together, Eqs. (1)–(6) and (16) completely describe the nonlinear dynamics of the film surface. These equations reduce to those studied in Ref. 22 if the anisotropy of the mobility is negligible.

III. LINEARIZED EQUATIONS OF MOTION

The equations of motion simplify considerably if the initial disturbance of the free surface of the metal film is small enough for the equations of motion to be linearized. To linearize, we put $h=h_0+h_1$ and $\Phi=-E_0x+\phi$, and work to first order in the small quantities h_1 and ϕ. Since the crystal structure is invariant under the reflection $y\to -y$, the equations of motion must be invariant under that transformation as well. It follows from Eq. (16) that

$$M_{12}(0) + M_{21}(0) = 0, \tag{17}$$

$$M_{11,2}(0) + M_{21,1}(0) = 0, \tag{18}$$

and

$$\frac{1}{q\nu_s\Omega}h_{1t} = M_{11}(0)\phi_{xx} + M_{22}(0)\phi_{yy} - E_0 M_{11,1}(0)h_{1xx} - E_0 M_{21,2}(0)h_{1yy} \tag{19}$$

for $z=h_0$. [Here $M_{ij,1}(h_x,h_y)$ denotes the partial derivative of $M_{ij}(h_x,h_y)$ with respect to h_x and $M_{ij,2}(h_x,h_y)$ is defined analogously.] Equations (1)–(3) yield

$$\phi_z = -E_0 h_{1x} \tag{20}$$

for $z=h_0$,

$$\phi_z = 0 \tag{21}$$

for $z=0$, and

$$\nabla^2\phi = 0 \tag{22}$$

for $0\le z\le h_0$.

We seek solutions to the linearized equations of motion [Eqs. (19)–(22)] with

$$h_1 = h_{10}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{23}$$

and

$$\phi = \phi_0(z)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{24}$$

where $\mathbf{r}\equiv x\hat{x}+y\hat{y}$, $\mathbf{k}\equiv k_x\hat{x}+k_y\hat{y}$, $\omega=\omega(\mathbf{k})$, and h_{10} is a constant. The Laplace equation [Eq. (22)] shows that $\phi_{0zz} - k^2\phi_0=0$ for $0\le z\le h_0$. Imposing the boundary condition (21), we see that $\phi_0=A\cosh(kz)$, where A is an arbitrary constant. Equation (20) gives

$$kA\sinh(h_0 k) = -ik_x E_0 h_{10}. \tag{25}$$

Inserting Eqs. (23) and (24) into Eq. (19) and using Eq. (25), we obtain the dispersion relation

$$-\frac{i\omega}{q\nu_s\Omega E_0}=\frac{ik_x\coth(h_0k)}{k}[M_{11}(0)k_x^2+M_{22}(0)k_y^2]+M_{11,1}(0)k_x^2+M_{21,2}(0)k_y^2. \tag{26}$$

Schimschak and Krug[11] linearized the equations of motion for the one-dimensional case in which h is independent of the traverse coordinate y. The dispersion relation (26) reduces to their result for $k_y=0$, as it should. As first noted by Krug and co-workers,[7,11] if $qM_{11,1}(0)$ is positive, there is an instability which leads to the growth of surface ripples with wave vector parallel to the applied electric field. The effects of surface self-diffusion must be taken into account if the wavelength of the fastest growing mode is to be determined.

Equation (26) shows that, even if $qM_{11,1}(0)$ is negative, there is an instability if $qM_{21,2}(0)>0$. This instability leads to the growth of surface ripples with wave vector *perpendicular* to the applied electric field. To understand the origin of this instability, consider a sinusoidal perturbation to the free surface with wave vector $\mathbf{k}=k_y\hat{y}$ and suppose that $qM_{21,2}(0)$ is positive. Neglecting the spatial variation in the electric field and working to first order in the perturbation, Eqs. (11) and (12) show that the surface atomic current is

$$\mathbf{J}=q\nu_sM_{11}(0)E_0\hat{x}+q\nu_sM_{21,2}(0)E_0h_y\mathbf{R}_y. \tag{27}$$

The first term on the right-hand side of Eq. (27) is independent of position and so does not affect the topography of the surface. On the other hand, since J_2 has the same sign as h_y, mass flows from the troughs to the crests of the sinusoidal disturbance. The disturbance therefore grows in amplitude as time passes, which means that there is an instability.

In this paper, we will restrict our attention to the time evolution of a stable surface. We will therefore assume that both $qM_{11,1}(0)$ and $qM_{21,2}(0)$ are negative.

IV. SHOCK WAVES

In this section, we will study shock waves on the metal film's free surface. Shock wave formation is an intrinsically nonlinear phenomenon, and so we must go beyond the linear theory we just developed.

A. Kinematic theory of one-dimensional shocks

We begin our analysis by developing a simple kinematic theory of one-dimensional shocks, i.e., shocks with a height h that is independent of y. This theory shows why shocks form and yields their velocity. Our treatment will be brief; a comprehensive discussion of the general kinematic theory of shock waves may be found in Ref. 25.

Suppose the film thickness varies very slowly with x. The surface atomic current $J=J_1$ on the film's free surface in a region of nearly uniform thickness h is approximately $q\nu_sM_{11}(0)E$, where E is the electric field strength in the region. The continuity equation for charge implies that $E_0h_0=Eh$, and so

$$J=J(h)=q\nu_sM_{11}(0)E_0h_0/h. \tag{28}$$

Since mass is conserved, $h_t+\Omega J_x=0$ or

$$h_t+c(h)h_x=0, \tag{29}$$

where $c(h)\equiv\Omega J'(h)=-q\nu_s\Omega M_{11}(0)E_0h_0/h^2$.

The velocity of a point on the film's surface of height h is $c(h)$. For the sake of specificity, we will assume that the effective charge q is negative for the remainder of this subsection; this assumption is valid for metals. The velocity $c(h)$ is then a decreasing function of h because $M_{11}(0)$ is positive. Equation (29) therefore predicts that a hump on the surface changes shape as it propagates—the right side of the hump broadens while the left side of the hump becomes steeper (see Fig. 2 of Ref. 24). Ultimately, the wave "breaks," rendering $h(x,t)$ a triple-valued function for a range of x values.

In reality, the surface atomic current J does not depend on h alone: Eq. (11) shows that it also depends on the spatial derivatives of h and ϕ. Equation (29) does not apply in regions in which h varies rapidly with x and, in fact, smoothing terms omitted in Eq. (28) ensure that breaking never occurs. We will return to this point later in the paper.

In a region in which h is an increasing function of x, a height discontinuity or shock will form in the limit in which the smoothing terms are vanishingly small.[25] Consider a shock that separates two regions in which h is constant. Let h_l (h_r) be the film thickness to the left (right) of the shock. For a shock, we must have $h_l<h_r$. The shock velocity is

$$U_0=\Omega\frac{J(h_l)-J(h_r)}{h_l-h_r} \tag{30}$$

in general.[25] In our problem, Eq. (28) applies and Eq. (30) becomes

$$U_0=-\frac{q\nu_s\Omega M_{11}(0)E_0h_0}{h_r\,h_l}=c_0\frac{h_0^2}{h_r\,h_l}, \tag{31}$$

where $c_0\equiv c(h_0)=-q\nu_s\Omega M_{11}(0)E_0/h_0>0$ is the phase velocity of surface waves propagating in the x direction in the limit of vanishing amplitude and wave number.[11,35]

As one would expect *a priori*, the shock velocity U_0 is proportional to $|q|$, $M_{11}(0)$, and E_0. Let $\langle h\rangle\equiv(h_r+h_l)/2$ be the average height and $\Delta h\equiv h_r-h_l$ be the height discontinuity at the shock. Then

$$U_0=c_0\frac{h_0^2}{\langle h\rangle^2-\frac{1}{4}(\Delta h)^2}, \tag{32}$$

which shows that U_0 is an increasing function of Δh and a decreasing function of $\langle h\rangle$.

B. Asymptotic analysis

The theory of one-dimensional shocks developed in Sec. IV A reveals nothing about the structure of a shock when smoothing effects are taken into account and does not apply when h depends upon the transverse coordinate y. We will now develop an approximate equation of motion which includes smoothing effects, and that applies when h varies slowly with y. This equation will allow us to study the structure of one-dimensional shocks and their interactions, as well as the propagation and structure of oblique shocks.

Consider the propagation of a localized disturbance with characteristic amplitude a that has characteristic widths l_x and l_y in the x and y directions, respectively. We wish to study the limit in which the amplitude is small, and in which the height h varies slowly with x and even more slowly with y. To do so, we will use multiple scale asymptotic analysis.[36] Put $h=h_0+a\zeta$, where $\zeta=\zeta(x,y,t)$ is of order unity. We will study the limit in which $\alpha\equiv a/h_0$, $\delta\equiv h_0/l_x$, and $\epsilon\equiv l_x/l_y$ are all small. More precisely, we shall consider the limit in which α, δ, and ϵ tend to zero but α/δ and $\alpha^{1/2}/\epsilon$ remain finite and nonzero. It is in this limit that the effects of nonlinearity, dissipation, and diffraction balance.

We will assume that $M_{11}(0)$ and $M_{22}(0)$ are distinct. The special case in which these two components of the mobility tensor happen to coincide must be treated separately and is beyond the scope of this paper. We also note that, since $M_{21}(0)=0$, Eq. (17) implies that $M_{12}(0)$ vanishes.

To simplify the description of the problem, we introduce the dimensionless quantities $\tilde{x}\equiv x/l_x$, $\tilde{y}\equiv y/l_y$, $\tilde{z}\equiv z/h_0$, $\tilde{t}\equiv[|q|\nu_s\Omega M_{11}(0)E_0/h_0 l_x]t$, and $\tilde{\phi}(\tilde{x},\tilde{y},\tilde{z},\tilde{t})\equiv\phi(x,y,z,t)/(E_0 l_x)$. (Recall that we set $\Phi=-E_0x+\phi$, so that $\nabla\phi\to 0$ for $x\to\pm\infty$.) Equation (1) becomes

$$\tilde{\phi}_{\tilde{z}\tilde{z}}+\delta^2(\tilde{\phi}_{\tilde{x}\tilde{x}}+\epsilon^2\tilde{\phi}_{\tilde{y}\tilde{y}})=0, \tag{33}$$

which applies for $0\le\tilde{z}\le 1+\alpha\zeta(\tilde{x},\tilde{y},\tilde{t})$, and all $\tilde{x}$, $\tilde{y}$, and $\tilde{t}$. The boundary conditions are

$$\tilde{\phi}_{\tilde{z}}=0 \quad \text{for } \tilde{z}=0, \tag{34}$$

$$\tilde{\phi}_{\tilde{z}}=\alpha\delta^2(\tilde{\phi}_{\tilde{x}}-1)\zeta_{\tilde{x}}+\alpha\delta^2\epsilon^2\tilde{\phi}_{\tilde{y}}\zeta_{\tilde{y}} \quad \text{for } \tilde{z}=1+\alpha\zeta(\tilde{x},\tilde{y},\tilde{t}), \tag{35}$$

$$\zeta\to 0 \quad \text{for } \tilde{x}\to\pm\infty, \tag{36}$$

and

$$\tilde{\phi}_{\tilde{x}}\hat{x}+\epsilon\tilde{\phi}_{\tilde{y}}\hat{y}\to 0 \quad \text{for } 0\le\tilde{z}\le 1 \text{ and } \tilde{x}\to\pm\infty. \tag{37}$$

Finally, the equation of motion for the free surface of the film is

$$\begin{aligned}\sigma_q\alpha\zeta_{\tilde{t}}=\partial_{\tilde{x}}\Bigg(&\frac{1}{\sqrt{g}}\{\tilde{M}_{11}[(1+\alpha^2\delta^2\epsilon^2\zeta_{\tilde{y}}^2)(\partial_{\tilde{x}}\tilde{\phi}-1)\\&-\alpha^2\delta^2\epsilon^2\zeta_{\tilde{x}}\zeta_{\tilde{y}}\partial_{\tilde{y}}\tilde{\phi}]\\&+\tilde{M}_{12}[(1+\alpha^2\delta^2\zeta_{\tilde{x}}^2)\epsilon\partial_{\tilde{y}}\tilde{\phi}-\alpha^2\delta^2\epsilon\zeta_{\tilde{x}}\zeta_{\tilde{y}}(\partial_{\tilde{x}}\tilde{\phi}-1)]\}\Bigg)\\&+\epsilon\partial_{\tilde{y}}\Bigg(\frac{1}{\sqrt{g}}\{\tilde{M}_{21}[(1+\alpha^2\delta^2\epsilon^2\zeta_{\tilde{y}}^2)(\partial_{\tilde{x}}\tilde{\phi}-1)\\&-\alpha^2\delta^2\epsilon^2\zeta_{\tilde{x}}\zeta_{\tilde{y}}\partial_{\tilde{y}}\tilde{\phi}]\\&+\tilde{M}_{22}[(1+\alpha^2\delta^2\zeta_{\tilde{x}}^2)\epsilon\partial_{\tilde{y}}\tilde{\phi}-\alpha^2\delta^2\epsilon\zeta_{\tilde{x}}\zeta_{\tilde{y}}(\partial_{\tilde{x}}\tilde{\phi}-1)]\}\Bigg),\end{aligned} \tag{38}$$

where $\tilde{M}_{ij}(h_x,h_y)\equiv M_{ij}(h_x,h_y)/M_{11}(0)$, $\tilde{z}=1+\alpha\zeta(\tilde{x},\tilde{y},\tilde{t})$, and $\sigma_q\equiv q/|q|$. Note that $g=1+\alpha^2\delta^2(\zeta_{\tilde{x}}^2+\epsilon^2\zeta_{\tilde{y}}^2)$.

We next introduce the scaled variables

$$\xi\equiv\frac{\alpha}{\delta}(\tilde{x}+\sigma_q\tilde{t}), \tag{39}$$

$$\eta\equiv\frac{\alpha}{\delta}\left(\frac{\alpha^{1/2}}{\epsilon}\right)\tilde{y}, \tag{40}$$

$$\tau\equiv\frac{\alpha^2}{\delta}\tilde{t}, \tag{41}$$

and

$$\psi\equiv\frac{1}{\delta}\tilde{\phi}. \tag{42}$$

$(\xi,\eta,\tilde{z})$ is a moving coordinate system that translates with velocity c_0 relative to the laboratory frame.

For convenience, we drop the tilde on z. Laplace's equation becomes

$$\psi_{zz}+\alpha^2\psi_{\xi\xi}+\alpha^3\psi_{\eta\eta}=0, \tag{43}$$

and this applies for $0\le z\le 1+\alpha\zeta(\xi,\eta,\tau)$, and all ξ, η, and τ. In terms of the scaled variables, Eqs. (34), (36), and (37) are

$$\psi_z=0 \quad \text{for } z=0, \tag{44}$$

$$\zeta\to 0 \quad \text{for } \xi\to\pm\infty, \tag{45}$$

and

$$\psi_\xi\to 0 \quad \text{and} \quad \psi_\eta\to 0 \quad \text{for } 0\le z\le 1 \text{ and } \xi\to\pm\infty. \tag{46}$$

After rewriting the free-surface boundary condition (35) in terms of the scaled variables and expanding to third order in α, we obtain

$$\psi_z+\alpha\zeta\psi_{zz}+\frac{1}{2}\alpha^2\zeta^2\psi_{zzz}+\frac{1}{6}\alpha^3\zeta^3\psi_{zzzz}=\alpha^3\psi_\xi\zeta_\xi-\alpha^2\zeta_\xi. \tag{47}$$

This result holds for $z=1$ and is valid to $O(\alpha^3)$.

We next turn our attention to the equation of motion (38). We rewrite this equation in terms of the scaled variables, expand it to second order in α, and use the fact that $M_{12}(0)=M_{21}(0)=0$. This shows that to order α and for $z=1$,

$$\zeta_\xi+\sigma_q\alpha\zeta_\tau=\psi_{\xi\xi}+\alpha\tilde{M}_{22}(0)\psi_{\eta\eta}-\alpha\tilde{M}_{11,1}(0)\zeta_{\xi\xi}+\alpha\partial_\xi^2(\zeta\psi_z), \tag{48}$$

where $\tilde{M}_{11,1}(h_x,h_y)\equiv M_{11,1}(h_x,h_y)/M_{11}(0)$. One advantage of introducing the scaled variables is now manifest: δ and ϵ do not appear explicitly in Eqs. (43)–(48).

We shall now begin our analysis of the small α limit in earnest. We assume that ζ and ψ can be expanded in powers of α for small α: we set

$$\zeta = \sum_{n=0}^{\infty} \alpha^n \zeta_n(\xi, \eta, \tau), \tag{49}$$

and

$$\psi = \sum_{n=0}^{\infty} \alpha^n \psi_n(\xi, \eta, \tau, z), \tag{50}$$

where the ζ_n's and ψ_n's are independent of α.

Our next step will be to insert the expansions (49) and (50) into Eqs. (43)–(48), and equate terms of the same order in α. The goal is to find a closed partial differential equation for ζ_0. To accomplish this, we will need to consider terms up to order α^3.

We begin by working to zeroth order in α. To this order, Eq. (43) becomes $\psi_{0zz}=0$ for $0 \le z \le 1$. This implies that ψ_{0z} does not depend on z. Applying Eq. (44), we see that in fact $\psi_{0z}=0$ for all z, ξ, η, and τ. We conclude that ψ_0 depends only on ξ, η, and τ, and we write

$$\psi_0 = \theta_0(\xi, \eta, \tau). \tag{51}$$

Equation (46) implies that

$$\theta_{0\xi} \to 0 \quad \text{and} \quad \theta_{0\eta} \to 0 \quad \text{for } \xi \to \pm\infty, \tag{52}$$

while, to zeroth order in α, the boundary condition (47) gives no new information.

Equation (45) shows that

$$\zeta_0 \to 0 \quad \text{for } \xi \to \pm\infty. \tag{53}$$

Inserting the expansions (49) and (50) into Eq. (48) and using Eq. (51), we find that $\zeta_{0\xi}=\theta_{0\xi\xi}$. We now integrate this result with respect to ξ. Since both ζ_0 and $\theta_{0\xi}$ vanish for $\xi \to \pm\infty$, the constant of integration is zero and

$$\zeta_0 = \theta_{0\xi}. \tag{54}$$

Working to first order in α, we see that ψ_1 is independent of z in much the same way as we showed that ψ_0 does not depend on z. We set

$$\psi_1 = \theta_1(\xi, \eta, \tau). \tag{55}$$

Equation (47) again yields no new information. Equation of motion (48) becomes

$$\zeta_{1\xi} + \sigma_q \zeta_{0\tau} = \theta_{1\xi\xi} - \tilde{M}_{11,1}(0)\zeta_{0\xi\xi} + \tilde{M}_{22}(0)\theta_{0\eta\eta}. \tag{56}$$

Our next task will be to write out the equations of motion to order α^2. Equations (43) and (51) show that

$$\psi_{2zz} = -\theta_{0\xi\xi} \tag{57}$$

for $0 \le z \le 1$. Integrating this with respect to z and using the boundary condition at the metal-insulator interface [Eq. (44)], we have $\psi_{2z}=-\theta_{0\xi\xi}z$. Integrating once again and using Eq. (54), we obtain

$$\psi_2 = \theta_2 - \frac{1}{2}\zeta_{0\xi} z^2, \tag{58}$$

where θ_2 depends only on ξ, η, and τ. Once more, Eq. (47) tells us nothing new.

Equating terms of order α^3 in Eq. (43) yields $\psi_{3zz}+\psi_{1\xi\xi}+\psi_{0\eta\eta}=0$ for $0 \le z \le 1$. Integrating this twice with respect to z and applying Eqs. (44), (51), and (55), we find that

$$\psi_3 = \theta_3 - \frac{1}{2}(\theta_{1\xi\xi} + \theta_{0\eta\eta})z^2, \tag{59}$$

where $0 \le z \le 1$ and $\theta_3 = \theta_3(\xi, \eta, \tau)$.

We conclude our analysis by equating terms of third order in Eq. (47). We obtain

$$\psi_{3z} + \zeta_0 \psi_{2zz} = \psi_{0\xi}\zeta_{0\xi} - \zeta_{1\xi} \tag{60}$$

for $z=1$. This can be simplified using Eqs. (54), (58), and (59). The result is

$$\theta_{1\xi\xi} + \theta_{0\eta\eta} = \zeta_{1\xi} - 2\zeta_0\zeta_{0\xi}. \tag{61}$$

Comparing this with Eq. (56), we have

$$\sigma_q\zeta_{0\tau} + 2\zeta_0\zeta_{0\xi} + \tilde{M}_{11,1}(0)\zeta_{0\xi\xi} = [\tilde{M}_{22}(0) - 1]\theta_{0\eta\eta}. \tag{62}$$

Taking the partial with respect to ξ of this equation and using Eq. (54) yields

$$[\sigma_q\zeta_{0\tau} + 2\zeta_0\zeta_{0\xi} + \tilde{M}_{11,1}(0)\zeta_{0\xi\xi}]_\xi = [\tilde{M}_{22}(0) - 1]\zeta_{0\eta\eta}. \tag{63}$$

The right-hand side of this equation is nonzero because we assumed that $M_{11}(0)$ and $M_{22}(0)$ differ. Therefore, to leading order, Eq. (63) includes the effect of transverse variations in the film height on the dynamics.

Equation (63) is a nonlinear partial differential equation which completely describes the dynamics of the film free surface in the small amplitude long-wavelength limit. This equation represents a considerable simplification of the problem: in the original formulation, $\Phi=\Phi(x,y,z,t)$ satisfies the three-dimensional Laplace equation and is subject to boundary conditions at a moving boundary, a stationary plane boundary, and at infinity. The motion of the free surface, which is described by the very complex equation of motion (16), is in turn dictated by the electrical potential at the film surface.

Before continuing, we shall simplify our notation. Let σ denote the sign of $q[M_{11}(0)-M_{22}(0)]$, and set $X=\sigma\xi$, $Y=|\tilde{M}_{22}(0)-1|^{-1/2}\eta$, $T=\tau$, $u=2\sigma\sigma_q\zeta_0$, and $\Psi=2\sigma_q\theta_0$. Equations (62), (54), and (63) become

$$u_T + uu_X - \nu u_{XX} + \Psi_{YY} = 0, \tag{64}$$

$$u = \Psi_X, \tag{65}$$

and

$$(u_T + uu_X - \nu u_{XX})_X + u_{YY} = 0, \tag{66}$$

where $\nu \equiv -\sigma_q\tilde{M}_{11,1}(0)$ is positive because $qM_{11,1}(0)$ was assumed to be negative.

Equation (66) includes the effects of nonlinearity, dissipation, and diffraction. The nonlinear term is present because the electrical current is crowded in regions in which the film thickness is reduced. The third term on the left-hand side is dissipative while the fourth term is diffractive. These two terms result from the crystalline anisotropy.

At the outset of our analysis, it was assumed that the applied electric current is large enough that the effects of SEM are much more important than those of capillarity. A simple dimensional analysis reveals that this assumption is valid if $l_x \gg l_E$, where the electromigration length $l_E \equiv \sqrt{\gamma_s\Omega/(|q|E_0)}$ and γ_s is the surface tension of the free surface of the metal.

SEM does not create or destroy atoms—it simply moves them across the surface of the solid. The total mass, which is proportional to $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(x,y,t)dxdy$, is therefore conserved. This conservation law was not lost in the asymptotic analysis. Indeed, Eq. (64) may be written

$$h_{1t} + \Omega\,\nabla\cdot\mathbf{J} = 0, \tag{67}$$

where

$$\Omega\mathbf{J} = \left[-c_0h_0 + c_0h_1 - \frac{c_0}{h_0}h_1^2 + q\nu_s\Omega M_{11,1}(0)E_0h_{1x}\right]\hat{x} + q\nu_s\Omega[M_{11}(0) - M_{22}(0)]\phi_y\hat{y}, \tag{68}$$

and we remind the reader that $h_1 \equiv h - h_0$. Equation (67) is the local statement of mass conservation. The first three terms on the right-hand side of Eq. (68) can be obtained from a Taylor's series expansion of the atomic mass current used in Sec. IV A [see Eq. (28)]. The fourth term on the right-hand side of Eq. (68) is a smoothing term that was omitted in the kinematic theory of one-dimensional shocks and prevents a wave from breaking. The final term results from the variation in the film thickness h with the transverse coordinate y.

Equation (66) may be written as

$$\partial_x\left\{h_{1t} + \partial_x\left[c_0h_1 - \frac{c_0}{h_0}h_1^2 + q\nu_s\Omega M_{11,1}(0)E_0h_{1x}\right]\right\} + \frac{q\nu_s\Omega E_0}{h_0}[M_{11}(0) - M_{22}(0)]h_{1yy} = 0. \tag{69}$$

Linearizing this equation and seeking solutions of the form $h_1 = h_{10}\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, we find that

$$-\frac{i\omega h_0}{q\nu_s\Omega E_0k_x} = iM_{11}(0) + M_{11,1}(0)h_0k_x + i[M_{22}(0) - M_{11}(0)]\frac{k_y^2}{k_x^2}. \tag{70}$$

On the other hand, for small h_0k_x and k_y/k_x, the linear dispersion relation (26) reduces to Eq. (70) if only terms of order h_0k_x and k_y^2/k_x^2 are retained. The result of the asymptotic analysis [Eq. (66)] and the linear theory therefore agree in the appropriate limit.

C. One-dimensional disturbances

If the initial surface height is independent of y, both h and Φ will remain independent of y for all subsequent times. u and Ψ are then independent of Y for all $T \geq 0$, and Eq. (64) becomes

$$u_T + uu_X = \nu u_{XX}, \tag{71}$$

which is the Burgers equation.

The Cole-Hopf transformation[37,38]

$$u = -2\nu\rho_X/\rho \tag{72}$$

reduces the Burgers equation [Eq. (71)] to the diffusion equation in one dimension,

$$\rho_T = \nu\rho_{XX}. \tag{73}$$

This transformation therefore converts the original nonlinear equation to an exactly solvable linear equation. The solution to the diffusion equation in one dimension is

$$\rho(X,T) = \frac{1}{\sqrt{4\pi\nu T}}\int_{-\infty}^{\infty}\rho(X',0)\exp\left(-\frac{(X-X')^2}{4\nu T}\right)dX'. \tag{74}$$

If $u_0(X) \equiv u(X,0)$ is given, then

$$\rho(X,0) = \exp\left(-\frac{1}{2\nu}\int_0^X u_0(X')dX'\right) \tag{75}$$

is known as well. Inserting this result into Eq. (74) and using Eq. (72), we find that

$$u(X,T) = \frac{\int_{-\infty}^{\infty}\frac{X-X'}{T}\exp\left(-\frac{1}{2\nu}G(X,X';T)\right)dX'}{\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2\nu}G(X,X';T)\right)dX'}, \tag{76}$$

where

$$G(X,X';T) \equiv \int_0^{X'} u_0(X'')dX'' + \frac{(X-X')^2}{2T}. \tag{77}$$

Equations (76) and (77) give the exact solution to the initial value problem for the Burgers equation.

A great deal is known about the Burgers equation in the $\nu \to 0^+$ limit because it has been studied extensively as a simplified model of turbulence. In this limit, a white-noise initial disturbance $u(X,0)$ evolves into a series of line segments of slope $1/T$ which meet at jump discontinuities, i.e., at shocks.[39,40] Suppose that a shock is located at the point $X = X_0$ at time T. The shock's velocity is $\frac{1}{2}[u(X=X_0^-,T) + u(X=X_0^+,T)]$. The strength of the shock, which is defined to be $u(X=X_0^-,T) - u(X=X_0^+,T)$, is always positive. Different shocks have different velocities, and so shocks can collide. When two shocks collide, they coalesce to form a single shock whose strength is the sum of the strengths of the two original shocks. The average shock strength and the interface width scale as $t^{-1/3}$ while the average distance between shocks scales as $t^{2/3}$ (see, for example, Ref. 40).

Is the $\nu \to 0^+$ limit relevant to our problem? The effective charge q is negative for metals and, for the sake of specificity, we will assume that $q < 0$ for the remainder of this subsection. If the undisturbed surface is close to a mobility maximum or minimum, then $\nu = \tilde{M}_{11,1}(0)$ will be small. Mobility maxima, however, are achieved for low index crystallographic planes, and faceting of the surface could occur as a consequence. Since we are employing a continuum description of the metal surface, our work does not apply to that case. We conclude that the limit $\nu \to 0^+$ is approached if the undisturbed surface is stable and has a mobility that is close to the minimum value.

For nonzero ν, there are no discontinuities and the shocks are rounded. The shock width tends to zero in the $\nu \to 0^+$ limit. For example, the Burgers equation [Eq. (71)] has the solution

$$u = \frac{u_l + u_r}{2} - \frac{u_l - u_r}{2} \tanh\left(\frac{u_l - u_r}{4\nu}(X - V_0 T)\right), \tag{78}$$

where u_l and u_r are constants, $u_l > u_r$, and $V_0 \equiv (u_l + u_r)/2$. The scaled surface displacement u asymptotes to u_l for $X \to -\infty$ and to u_r for $X \to +\infty$. This solution is a shock wave which propagates with velocity V_0; the width of the shock $w_0 = 4\nu/(u_l - u_r)$ tends to zero as ν becomes small. The shock has a steady-state form because the tendency for the wave to steepen and break is counterbalanced by the diffusive term in the Burgers equation.

In the laboratory frame of reference (x, y, t), the velocity of the shock (78) is

$$U_0 = c_0\left(1 - \frac{h_{1l} + h_{1r}}{h_0}\right), \tag{79}$$

where $h_{1l} \equiv h_l - h_0$ and $h_{1r} \equiv h_r - h_0$. The shock velocity given by the kinematic analysis [Eq. (31)] reduces to this result if h_{1l} and h_{1r} are small. The shock width was not revealed by the kinematic analysis; Eq. (78) shows that it is $2|\tilde{M}_{11,1}(0)|h_0^2/(h_r - h_l)$ in the original units.

D. Two-dimensional disturbances

For initial disturbances that depend on both x and y, the equation of motion is Eq. (66). This equation occurs in nonlinear acoustics[30,31] and in the study of Mach reflection of weak shocks,[41] and is known as the DZK equation. The DZK equation cannot be solved using a Cole-Hopf transformation. Only a few exact solutions are known but these have shed light on caustic and shock formation in converging nonlinear waves.[42,43] Numerical integration of the DZK equation is challenging but a variety of methods of increasing speed and reliability have been developed.[44]

Quasi-one-dimensional solutions which propagate at an angle γ to the X axis are readily obtained. We seek solutions to the DZK equation which depend only on T and $\tilde{X} \equiv X + Y \tan \gamma - T \tan^2 \gamma$. The DZK equation then reduces to

$$u_T + u u_{\tilde{X}} - \nu u_{\tilde{X}\tilde{X}} = 0, \tag{80}$$

which is just the Burgers equation. Equation (78) with X replaced by $\tilde{X}$ is a solution to Eq. (80). Thus, the DZK equation has the solution

$$u = \frac{u_l + u_r}{2} - \frac{u_l - u_r}{2} \tanh\left(\frac{u_l - u_r}{4\nu}[X + Y \tan \gamma - V(\gamma)T]\right), \tag{81}$$

where u_l and u_r are constants, $u_l > u_r$, and

$$V(\gamma) \equiv \frac{u_l + u_r}{2} + \tan^2 \gamma. \tag{82}$$

Note that $u \to u_l$ for $X \to -\infty$ and $u \to u_r$ for $X \to \infty$. The solution (81) is an oblique shock with contour lines that make an angle γ with the Y axis; this solution reduces to the one-dimensional shock (78) for $\gamma = 0$.

In the laboratory frame of reference (x, y, t), Eq. (81) is

$$h = \frac{h_l + h_r}{2} - \frac{h_l - h_r}{2} \tanh\left(\frac{\hat{\mathbf{e}} \cdot \mathbf{r} - U(\theta)t}{w(\theta)}\right), \tag{83}$$

where the shock's direction of propagation $\hat{\mathbf{e}} \equiv \hat{x} \cos \theta + \hat{y} \sin \theta$ makes an angle θ with the x axis,

$$U(\theta) = c_0\left(1 - \frac{h_{1l} + h_{1r}}{h_0} - \frac{M_{11}(0) - M_{22}(0)}{M_{11}(0)} \tan^2 \theta\right) \cos \theta \tag{84}$$

is the shock velocity, and

$$w(\theta) = 2 \cos \theta |\tilde{M}_{11,1}(0)| \frac{h_0^2}{|h_r - h_l|} \tag{85}$$

is the width of the shock. The angles θ and γ do not coincide because x and y were scaled in different ways to obtain X and Y. Explicitly,

$$\tan \theta = \sigma |\tilde{M}_{22}(0) - 1|^{-1/2} \alpha^{1/2} \tan \gamma, \tag{86}$$

which shows that θ must be small for Eqs. (83)–(85) to apply.

Some discussion of these results is in order. Naturally, $U(\theta)$ and $w(\theta)$ reduce to U_0 and w_0 for $\theta = 0$, and so Eqs. (84) and (85) generalize the results of Sec. IV C. Equation (70), the linear dispersion relation that applies for $h_0 k_x \ll 1$ and $k_y \ll k_x$, yields the phase velocity

$$v_p(\theta) = c_0\left(1 - \frac{M_{11}(0) - M_{22}(0)}{M_{11}(0)} \tan^2 \theta\right) \cos \theta \tag{87}$$

for a wave traveling in the direction $\hat{\mathbf{e}}$. This shows that the shock velocity $U(\theta)$ reduces to the phase velocity of a long-wavelength small amplitude wave $v_p(\theta)$ in the limit that h_r and h_l tend to h_0, as one would expect *a priori*. The shock width $w(\theta)$ is inversely proportional to the shock strength $|h_r - h_l|$, which means that strong shocks are narrow shocks.

V. CONCLUSIONS

We have shown that surface electromigration can produce shock waves on the surface of a current-carrying single-crystal miscut metal thin film when it is disturbed. These shocks form because $c(h)$, the velocity of a point on the film's surface of height h, depends on h. Shock waves are an inherently nonlinear phenomenon; our problem is nonlinear because the electrical current is crowded in regions in which the film thickness is reduced.

Shocks form on stable surfaces. It has been known for some time that the planar free surface of the film is unstable if $qM_{11,1}(0)E_0$ is positive. Our linear stability analysis reveals that, even if $qM_{11,1}(0)E_0$ is negative, the surface is unstable if $qM_{21,2}(0)E_0$ is positive. In this case, the instability leads to the growth of surface ripples with their wave vector perpendicular to the applied electric field. Both $qM_{11,1}(0)E_0$ and $qM_{21,2}(0)E_0$ must be negative if the surface is to be stable.

To gain insight into shock structure and propagation, we derived an approximate equation of motion using multiple scale asymptotic analysis. This equation of motion applies to surface disturbances that have small amplitude, that vary slowly with the longitudinal coordinate x, and that vary more slowly still with the transverse coordinate y.

For disturbances which do not depend on y, the approximate equation of motion reduces to the Burgers equation $u_t+uu_x=\nu u_{xx}$. Shocks that preserve their form as they propagate can be formed. In these shocks, the tendency of the wave to steepen, which is caused by the nonlinearity, is counterbalanced by the smoothing effect of the anisotropic adatom mobility.

If the undisturbed surface has a mobility that is close to the minimum value, ν is small and a white-noise initial disturbance evolves into a series of line segments which meet at shocks. When two shocks collide, they coalesce to form a single shock whose strength is the sum of the strengths of the two original shocks. As a result, the average distance between shocks increases as time passes. Obviously, shock coalescence also increases the average shock strength. However, the slope of the line segments between shocks is proportional to $1/t$, and this leads to an overall decline in the average shock strength with the passage of time. Detailed studies of the Burgers equation[39,40] have shown that the average distance between shocks scales as $t^{2/3}$ and that the average shock strength scales as $t^{-1/3}$ in the so-called low viscosity limit $\nu\to 0^+$.

In the general case in which h depends on both x and y, the approximate equation of motion is the dissipative Zabolotskaya-Khokhlov equation. The velocity and width of a shock whose direction of propagation makes a small angle with the applied electric field was determined. The long-time behavior of solutions to the DZK equation with white-noise initial conditions has not yet been studied, and would be an interesting (albeit very challenging) topic for future research.

Electromigration-induced shock wave propagation has not yet been observed experimentally. For it to be observed, a single-crystal metal thin film should first be prepared. The film must be oriented so that its planar free surface would be stable if an electric field were applied in the x direction. After a step whose height increases with x is etched into the surface, a high electric field should be applied in the x direction. The step will then evolve into a shock wave that propagates along the direction of the applied field and that has a steady-state tanh profile with width inversely proportional to its height. If two or more parallel steps are etched into the surface initially, applying the electric field will lead to the collision and fusion of shocks.

[1] S. Stoyanov, Jpn. J. Appl. Phys., Part 1 **30**, 1 (1991).
[2] D. Kandel and J. D. Weeks, Phys. Rev. Lett. **74**, 3632 (1995).
[3] D. J. Liu and J. D. Weeks, Phys. Rev. B **57**, 14891 (1998).
[4] J. Chang, O. Pierre-Louis, and C. Misbah, Phys. Rev. Lett. **96**, 195901 (2006).
[5] V. Popkov and J. Krug, Phys. Rev. B **73**, 235430 (2006).
[6] M. Dufay, T. Frisch, and J. M. Debierre, Phys. Rev. B **75**, 241304(R) (2007).
[7] J. Krug and H. T. Dobbs, Phys. Rev. Lett. **73**, 1947 (1994).
[8] O. Kraft and E. Arzt, Appl. Phys. Lett. **66**, 2063 (1995); Acta Mater. **45**, 1599 (1997).
[9] W. Wang, Z. Suo, and T. H. Hao, J. Appl. Phys. **79**, 2394 (1996).
[10] M. Mahadevan and R. M. Bradley, J. Appl. Phys. **79**, 6840 (1996).
[11] M. Schimschak and J. Krug, Phys. Rev. Lett. **78**, 278 (1997).
[12] L. Xia, A. F. Bower, Z. Suo, and C. F. Shih, J. Mech. Phys. Solids **45**, 1473 (1997).
[13] M. Schimschak and J. Krug, Phys. Rev. Lett. **80**, 1674 (1998).
[14] M. R. Gungor and D. Maroudas, Appl. Phys. Lett. **72**, 3452 (1998).
[15] M. R. Gungor and D. Maroudas, Surf. Sci. **415**, L1055 (1998).
[16] M. R. Gungor and D. Maroudas, J. Appl. Phys. **85**, 2233 (1999).
[17] M. Mahadevan, R. M. Bradley, and J.-M. Debierre, Europhys. Lett. **45**, 680 (1999).
[18] M. Mahadevan and R. M. Bradley, Phys. Rev. B **59**, 11037 (1999).
[19] D. R. Fridline and A. F. Bower, J. Appl. Phys. **85**, 3168 (1999).
[20] R. M. Bradley, Phys. Rev. E **60**, 3736 (1999).
[21] M. Schimschak and J. Krug, J. Appl. Phys. **87**, 695 (2000).
[22] R. M. Bradley, Physica D **158**, 216 (2001).
[23] R. M. Bradley, Phys. Rev. E **65**, 036603 (2002).
[24] R. M. Bradley, Appl. Phys. Lett. **93**, 213105 (2008).
[25] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).
[26] A. V. Latyshev, A. L. Aseev, A. B. Krasilnikov, and S. I. Stenin, Surf. Sci. **213**, 157 (1989).
[27] M. J. Ramstad, R. J. Birgeneau, K. I. Blum, D. Y. Noh, B. O. Wells, and M. J. Young, Europhys. Lett. **24**, 653 (1993).
[28] A brief account of some of the results on one-dimensional disturbances that are obtained in this paper appears in Ref. 24.
[29] For applications of the Burgers equation, see A. Noullez, S. N. Gurbatov, E. Aurell, and S. I. Simdyankin, Phys. Rev. E **71**, 056305 (2005), and references therein.
[30] E. A. Zabolotskaya and R. V. Khokhlov, Sov. Phys. Acoust. **15**, 35 (1969).
[31] V. P. Kuznetsov, Sov. Phys. Acoust. **16**, 467 (1971).
[32] M. Bartuccelli, P. Pantano, and T. Brugarino, Lett. Nuovo Cimento Soc. Ital. Fis. **37**, 433 (1983).
[33] We assume that the electrical conductivity is isotropic, which is an excellent approximation for nearly all metals.
[34] J. Jost, *Riemannian Geometry and Geometric Analysis*, 3rd ed. (Springer, Berlin, 2002), p. 85.
[35] If the effective charge q is positive, a shock has $h_l>h_r$ and Eq. (31) continues to apply.

[36] P. G. Drazin and R. S. Johnson, *Solitons: An Introduction* (Cambridge University, Cambridge, 1989).
[37] E. Hopf, Commun. Pure Appl. Math. **3**, 201 (1950).
[38] J. D. Cole, Q. Appl. Math. **9**, 225 (1951).
[39] J. M. Burgers, *The Nonlinear Diffusion Equation* (Riedel, Dordrecht, 1974).
[40] L. Frachebourg and P. A. Martin, J. Fluid Mech. **417**, 323 (2000).
[41] M. Brio and J. K. Hunter, Physica D **60**, 194 (1992).
[42] A. T. Cates and D. G. Crighton, Proc. R. Soc. London, Ser. A **430**, 69 (1990).
[43] P. Sionoid and A. T. Cates, Proc. R. Soc. London, Ser. A **447**, 253 (1994).
[44] See C. Tao, J. Ma, Z. Zhu, G. Du, and Z. Ping, J. Acoust. Soc. Am. **114**, 114 (2003), and references therein.